OCR
RECOGNISING ACHIEVEMENT

Oxford, Cambridge and RSA Examinations

Advanced Subsidiary General Certificate of Education
Advanced General Certificate of Education

MEI STRUCTURED MATHEMATICS 2610/1

Differential Equations (Mechanics 4)

Monday **14 JANUARY 2002** Afternoon 1 hour 20 minutes

Additional materials:
Answer booklet
Graph paper
MEI Examination Formulae and Tables (MF12)

TIME 1 hour 20 minutes

INSTRUCTIONS TO CANDIDATES

- Write your Name, Centre Number and Candidate Number in the spaces provided on the answer booklet.
- Answer any **three** questions.
- You are permitted to use a graphical calculator in this paper.
- There is an insert for use with Question **3**.

INFORMATION FOR CANDIDATES

- The approximate allocation of marks is given in brackets [ ] at the end of each question or part question.
- You are advised that an answer may receive no marks unless you show sufficient detail of the working to indicate that a correct method is being used.
- Final answers should be given to a degree of accuracy appropriate to the context.
- Take $g = 9.8 \text{ m s}^{-2}$ unless otherwise instructed.
- The total number of marks for this paper is 60.

This question paper consists of 3 printed pages, 1 blank page and an insert.

HN/3

[Turn over

1 The differential equation,

$$\frac{dy}{dx} + \frac{y}{1+x} = x^{\frac{3}{2}} \qquad x \geqslant 0$$

is to be solved.

(i) Find the particular solution subject to the initial condition $x = 0$, $y = 2$. Describe the behaviour of the solution as x tends to infinity. [9]

A more general case of the above equation is

$$(1+x)\frac{dy}{dx} + ny = x^{\frac{3}{2}}(1+x)^{2-n} \qquad x \geqslant 0,$$

where n is a real number.

(ii) Solve the differential equation to show that the general solution is

$$y = \frac{2x^{\frac{5}{2}}(7+5x)+C}{35(1+x)^n},$$

where C is an arbitrary constant. [5]

Describe the behaviour of the general solution as x tends to infinity, identifying three distinct cases and the values of n for which they occur. [6]

2 A chemical reaction takes place in a chamber. Compound X is injected into the chamber at a constant rate and compound Y (which is produced in the reaction) is extracted at a rate proportional to the quantity of compound Y in the chamber. The equations modelling the system are

$$\frac{dx}{dt} = -4x + y + 28, \qquad \frac{dy}{dt} = 6x - 5y,$$

where x grams and y grams are the quantities in the chamber of compounds X and Y respectively, and t is the time in minutes.

(i) Eliminate y from the equations to show that

$$\frac{d^2x}{dt^2} + 9\frac{dx}{dt} + 14x = 140. \qquad [5]$$

Initially there is no X and there is no Y in the chamber.

(ii) Solve the differential equation to find x in terms of t. Hence find y in terms of t. [10]

(iii) Sketch a graph of the solutions, indicating the values of x and y after a long period of time.

Explain clearly how, if it is known that such long-term values exist, they can be calculated without finding the solutions for x and y in terms of t. [5]

3 [There is an insert for use in parts **(iii)** and **(iv)** of this question.]

The differential equation

$$\frac{dy}{dx} = \frac{y^2 - 4}{(1 + x^{\frac{3}{4}})y}$$

is to be investigated using the tangent field.

(i) State the values of y for which the tangents (direction indicators) are

(A) parallel to the x-axis,
(B) parallel to the y-axis. [3]

(ii) Show that the x-axis is a line of symmetry for the tangent field. [1]

(iii) On the insert complete the table of gradients, giving all answers correct to 1 decimal place. Sketch the tangent field for the differential equation for $0 \leqslant x \leqslant 2$, $-2 \leqslant y \leqslant 2$. [8]

(iv) Sketch the solution curve starting at the point (0, –1) and the solution curve starting at the point (0, –1.5). [3]

(v) What does the tangent field suggest about the equation of the solution curve starting from (0, –2)? Verify that this equation is the particular solution for this initial condition. [2]

(vi) A particular solution curve starts from the point (0, k), where $0 < |k| < 2$, and x is increasing initially. Describe briefly the behaviour of the solution curve, stating the coordinates of the point at which it stops. Why does the solution curve stop at this point? [3]

4 The current, I amperes, in an electrical circuit at time t seconds after being switched on is modelled by the differential equation

$$\frac{d^2I}{dt^2} + 4\frac{dI}{dt} + 4I = -5\sin t.$$

(i) Find the general solution. [8]

Initially there is no current flowing but $\frac{dI}{dt} = 5$.

(ii) Show that the solution is

$$I = \tfrac{4}{5}\cos t - \tfrac{3}{5}\sin t + (4t - \tfrac{4}{5})e^{-2t}.$$ [5]

The exponentially decaying terms of the solution form the transient current. The non-decaying term forms the steady state current.

(iii) Calculate the amplitude of the steady state current. Express the steady state current in the form $R\cos(t + \alpha)$, calculating α correct to 4 significant figures. [3]

(iv) Calculate the maximum value of the transient current. What is the value of I at this time? [4]

Candidate Name	Centre Number	Candidate Number

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Differential Equations (Mechanics 4)

INSERT

Monday **14 JANUARY 2002** Afternoon 1 hour 20 minutes

INSTRUCTIONS TO CANDIDATES

- This insert should be used in Question **3**.
- Write your Name, Centre Number and Candidate Number in the spaces provided at the top of this page.
- If you have answered Question **3**, attach the insert securely to your answer booklet.

This insert consists of 2 printed pages.

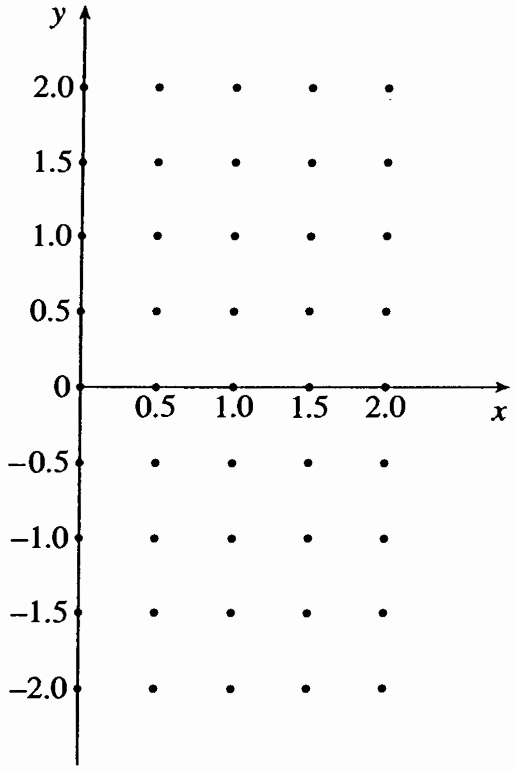
3 **(iii)**

Gradients

1.5					
1.0					
0.5					
y \ x	0	0.5	1.0	1.5	2.0

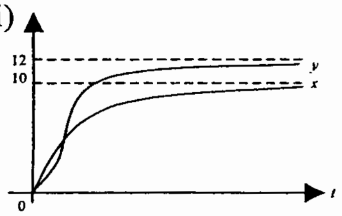
(iv) Sketch the solution.

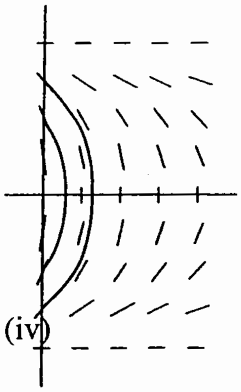
Tangent field



Mark Scheme

1(i)	$I = \exp(\int \frac{1}{1+x} dx) = \exp(\ln(1+x))$	M1	attempt integrating factor	
	$= 1+x$	A1		
	$\frac{d}{dx}(y(1+x)) = x^{3/2}(1+x)$	M1	multiply and recognise exact form	
	$y(1+x) = \int (x^{3/2} + x^{5/2}) dx$	M1	integrate	
	$= \frac{2}{5}x^{5/2} + \frac{2}{7}x^{7/2} + A$	A1		
	$x = 0, y = 2 \Rightarrow A = 2$	M1	boundary conditions	
		A1	arbitrary constant	
	$y = \dfrac{\frac{2}{5}x^{5/2} + \frac{2}{7}x^{7/2} + 2}{1+x}$	A1		
	$y \to \infty$ as $x \to \infty$	B1		9
(ii)	$I = \exp(\int \frac{n}{1+x} dx) = \exp(n \ln(1+x))$	M1	attempt integrating factor	
	$= (1+x)^n$	A1		
	$(1+x)^n \frac{dy}{dx} + n(1+x)^{n-1} y = x^{3/2}(1+x)$	M1	multiply and simplify RHS	
	$y(1+x)^n = \frac{2}{5}x^{5/2} + \frac{2}{7}x^{7/2} + B$	M1	integrate	
	$y = \dfrac{2x^{5/2}(7+5x) + C}{35(1+x)^n}$	E1		5
(iii)	For large x, $y \approx \dfrac{2x^{7/2}}{7(1+x)^n}$			
	y diverges	B1	Recognise for at least one correct value of n	
	for $n < \frac{7}{2}$	B1		
	$y \to 0$	B1	Recognise for at least one correct value of n	
	for $n > \frac{7}{2}$	B1		
	$y \to \frac{2}{7}$	B1		
	for $n = \frac{7}{2}$	B1		6

2(i)	$y = \frac{dx}{dt} + 4x - 28$	M1	y in terms of x and dx/dt	
	$\frac{dy}{dt} = \frac{d^2x}{dt^2} + 4\frac{dx}{dt}$	M1	differentiate	
	$6x - 5y = \frac{d^2x}{dt^2} + 4\frac{dx}{dt}$	M1	substitute for dy/dt	
	$6x - 5(\frac{dx}{dt} + 4x - 28) = \frac{d^2x}{dt^2} + 4\frac{dx}{dt}$	M1	substitute for y	
	$\frac{d^2x}{dt^2} + 9\frac{dx}{dt} + 14x = 140$	E1		5
(ii)	$\alpha^2 + 9\alpha + 14 = 0$	M1	auxiliary equation	
	$\alpha = -7$ or -2	A1		
	CF: $x = Ae^{-7t} + Be^{-2t}$	F1	complementary function	
	PI: $x = \frac{140}{14} = 10$	B1	particular integral	
	$x = 10 + Ae^{-7t} + Be^{-2t}$	F1	general solution	
	$y = \frac{dx}{dt} + 4x - 28 = 12 - 3Ae^{-7t} + 2Be^{-2t}$	M1	attempt to find y using correct arbitrary constants	
	$t = 0, x = 0 \Rightarrow 10 + A + B = 0$	M1	condition on x	
	$t = 0, y = 0 \Rightarrow 12 - 3A + 2B = 0$	M1	condition on y (or dx/dt)	
	[or $t = 0, \frac{dx}{dt} = 28 \Rightarrow -7A - 2B = 28$]			
	$\Rightarrow A = -\frac{8}{5}, B = -\frac{42}{5}$			
	$x = 10 - \frac{8}{5}e^{-7t} - \frac{42}{5}e^{-2t}$	A1		
	$y = 12 + \frac{24}{5}e^{-7t} - \frac{84}{5}e^{-2t}$	A1		10
(iii)		B1 B1 B1	x y long-term values clearly indicated both graphs must start at 0 and increase to horizontal asymptote deduct 1 for right idea but poor accuracy	
	if long-term values then rate of change zero (equilibium values)	B1		
	so set $\frac{dx}{dt}$ and $\frac{dy}{dt}$ to zero and solve for x and y	B1		5

3(i) (A) $y = \pm 2$	B1		
	B1		
(B) $y = 0$	B1		3
(ii) $\frac{dy}{dx}(x,-y) = -\frac{dy}{dx}(x,y) \Rightarrow$ gradient reflected	B1		1
(iii)	M1	Attempt to calculate gradients	
−1.2, −0.7, −0.6, −0.5, −0.4	A1		
−3.0, −1.9, −1.5, −1.3, −1.1	A1		
−7.5, −4.7, −3.8, −3.2, −2.8	A1		
	F1	vertical for $y = 0$	
	F1	horizontal for $y = \pm 2$	
	M1	attempt remainder	
	F1	remaining direction lines	8
(iv)	M1	attempt curves	
	F1	curve from −1.5	
	F1	curve from −1	3
(v) solution $y = -2$	B1		
$y = -2 \Rightarrow \frac{dy}{dx} = 0$	B1	verify satisfies DE	
$y = -2 \Rightarrow \frac{y^2 - 4}{(1 + x^{3/4})y} = \frac{(-2)^2 - 4}{(1 + x^{3/4})(-2)} = 0$			
passes through (0,−2)			2
(vi) x increases until curve reaches x-axis	B1		
then decreases until $(0,-k)$	B1		
stops as $\frac{dy}{dx}$ does not exist for $x < 0$	B1		3

4(i)	$\alpha^2+4\alpha+4=0$	M1	auxiliary equation	
	$\alpha=-2$	A1		
	CF: $I=(A+Bt)e^{-2t}$	F1	complementary function for repeated root	
	$I=a\sin t+b\cos t$	B1	trial solution	
	$\dot{I}=a\cos t-b\sin t,\ \ddot{I}=-a\sin t-b\cos t$	M1	differentiate twice	
	$\Rightarrow(4a-4b-a)\sin t+(4b+4a-b)\cos t=-5\sin t$	M1	substitute and compare coefficients	
	$3a-4b=-5$			
	$4a+3b=0$			
	$\Rightarrow a=-\frac{3}{5},\ b=\frac{4}{5}$	A1		
	$I=-\frac{3}{5}\sin t+\frac{4}{5}\cos t+(A+Bt)e^{-2t}$	F1	general solution	8
(ii)	$\frac{4}{5}+A=0$	M1	condition on I	
	$\dot{I}=-\frac{3}{5}\cos t-\frac{4}{5}\sin t+(-2A-2Bt+B)e^{-2t}$	M1	differentiate	
	$5=-\frac{3}{5}-2A+B$	M1	condition on $\dot{I}$	
	$A=-\frac{4}{5},B=4$	E1		
	$I=-\frac{3}{5}\sin t+\frac{4}{5}\cos t+(4t-\frac{4}{5})e^{-2t}$	E1		5
(iii)	$I_{SS}=-\frac{3}{5}\sin t+\frac{4}{5}\cos t\Rightarrow\text{amp.}=\sqrt{(-\frac{3}{5})^2+(\frac{4}{5})^2}=1$	B1		
	$R\cos(t+\alpha)=-\frac{3}{5}\sin t+\frac{4}{5}\cos t\Rightarrow R=1,\alpha=0.6435$	M1	attempt to calculate α	
	$I_{SS}=\cos(t+0.6435)$	A1	cao	3
(iv)	$I_T=(4t-\frac{4}{5})e^{-2t}$			
	$\dot{I}_T=(-8t+\frac{8}{5}+4)e^{-2t}$	M1	identify transient current and differentiate	
	$\dot{I}_T=0$ when $t=\frac{7}{10}$	A1		
	maximum value of $I_T=0.493$	A1		
	maximum value of $I=0.719$	A1		4

Examiner’s Report

Differential Equations (Mechanics 4) (2610)

General Comments

There were many very good scripts in this small entry. Candidates generally selected the correct methods to solve the differential equations and the levels of accuracy were an improvement on previous examinations. Questions 1, 2 and 4 were all popular, especially question 2 which was rarely omitted. Question 3 was avoided by most candidates.

Comments on Individual Questions

Q1 (Integrating Factor)

The first part was often done well with few candidates unable to find the integrating factor. Some candidates made slips in their integration but many correctly found the solution. Most candidates correctly stated that y tends to infinity but some gave an asymptotic curve which was not required. In the second part more candidates had difficulty in finding the integrating factor, but most were successful. Errors in algebraic manipulation hampered some candidates.

[(i) $y = \dfrac{\frac{2}{5}x^{\frac{5}{2}} + \frac{2}{7}x^{\frac{7}{2}} + 2}{1+x}$; $y \to \infty$ as $x \to \infty$; (ii) y diverges for $n < \frac{7}{2}$, $y \to 0$ for $n > \frac{7}{2}$, $y \to \frac{2}{7}$ for $n = \frac{7}{2}$]

Q2 (Simultaneous Equations)

The first part was well done but a surprising number of candidates 'differentiated' 28 with respect to t and obtained $28x$. The solution of the differential equation was well done although some made heavy work of finding the particular integral. It was pleasing to see the vast majority of candidates using the solution for x and the original equation for dx/dt to find y, although there were still a few candidates who tried to produce and solve a second order equation for y, with the usual problems of relating the two sets of arbitrary constants.

Most candidates knew roughly how to draw the graphs but there were some sloppy efforts and indications of long-term values were not always clear.

[(ii) $x = 10 - \frac{8}{5}e^{-7t} - \frac{42}{5}e^{-2t}$, $y = 12 + \frac{24}{5}e^{-7t} - \frac{84}{5}e^{-2t}$]

Q3 (Tangent Field)

This was not a popular question but virtually all who did it scored high marks. A few candidates made errors in their calculations of the gradients. A few candidates did not draw all the direction lines on the tangent field and some drew curves inconsistent with their tangent field. Most candidates realised what the solution in part (v) was but did not always state it clearly or show that it satisfied the differential equation. When describing a solution curve in the last part, some candidates were vague and/or ambiguous. Few candidates commented on why the curve stops.

[(i) (*A*) $y = \pm 2$, (*B*) $y = 0$; (v) $y = -2$; (vi) $(0, -k)$]

Q4 (Second Order Equation)

The first two parts of the question were often very well done with only minor slips. There were some good attempts to the remaining parts of the question, but answers were not always clearly stated in part (iii) and arithmetical slips were common in part (iv).

[(i) $I = -\frac{3}{5}\sin t + \frac{4}{5}\cos t + (A + Bt)e^{-2t}$; (iii) amplitude 1, $I_{ss} = \cos(t + 0.6435)$ (4 s. f.);
(iv) max $I_T = 0.493$ (3 s. f.), max $I = 0.719$ (3 s. f.)]